2 Polynomials

Lesson at a Glance

1. In a polynomial $p(x)$, the highest exponent of x is called the degree of the polynomial.
2. Polynomials of degrees 1, 2 and 3 are called linear, quadratic and cubic polynomials respectively.
3. If $p(x)$ is a polynomial in x, and if k is any real number, then the value obtained by replacing x by k in $p(x)$, is called the value of $p(x)$ at $x = k$, and is denoted by $p(k)$.
4. If on substituting $x = k$ in a polynomial $p(x)$, we get $p(k) = 0$, then k is said to be a zero of the polynomial.
5. Every real number is a constant polynomial.
6. 0 is the zero polynomial.
7. The degree of a non-zero constant polynomial is zero.
8. Polynomials of one term, two terms and three terms are called monomial, binomial and trinomial respectively.
9. $2x^3 + 5x^2 - 7x + \sqrt{3}$ is a polynomial in the variable x of degree 3.
10. $x^{5/2} + x^2 - 7x + 3$ is not a polynomial.
11. If the graph of a polynomial intersects x-axis at n points, then the number of zeroes of the polynomial is n.
12. If a linear polynomial is $p(x) = ax + b$, then zero of the polynomial

$$= \frac{-\text{(Constant term)}}{\text{Coefficient of } x} = \frac{-b}{a}.$$

13. If a quadratic polynomial is $p(x) = ax^2 + bx + c$, then

$$\text{Sum of zeroes} = \frac{-\text{(Coefficient of } x)}{\text{Coefficient of } x^2} = \frac{-b}{a}$$

$$\text{Product of zeroes} = \frac{\text{Constant term}}{\text{Coefficient of } x^2} = \frac{c}{a}.$$

14. If a cubic polynomial is $p(x) = ax^3 + bx^2 + cx + d$, then

$$\text{Sum of zeroes} = \frac{-(\text{Coefficient of } x^2)}{\text{Coefficient of } x^3} = \frac{-b}{a}$$

Sum of the product of zeroes taken two at a time

$$= \frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = \frac{c}{a}.$$

$$\text{Product of zeroes} = \frac{-(\text{Constant term})}{\text{Coefficient of } x^2} = \frac{-d}{a}.$$

15. If one polynomial $p(x)$ is divided by the other polynomial $g(x) \neq 0$, then the relation among $p(x)$, $g(x)$, quotient $q(x)$ and remainder $r(x)$ is given by

$p(x) = g(x) \times q(x) + r(x)$, where degree of $r(x)$ < degree of $g(x)$.

i.e., Dividend = Divisor × Quotient + Remainder

16. A linear polynomial has at most 1 zero.

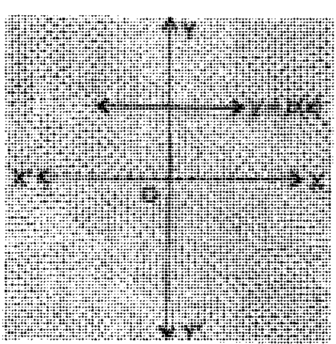
17. A quadratic polynomial has at most 2 zeroes.

18. A cubic polynomial has at most 3 zeroes.

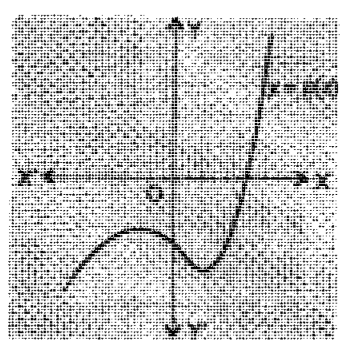
TEXTBOOK QUESTIONS SOLVED

Exercise 2.1 *(Page – 28)*

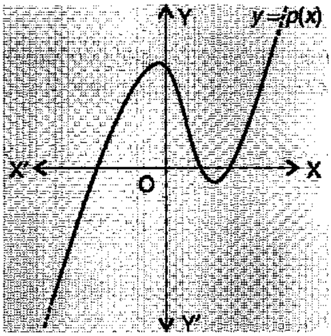
1. *The graphs of $y = p(x)$ are given in figure below, for some polynomials $p(x)$. Find the number of zeroes of $p(x)$, in each case.*



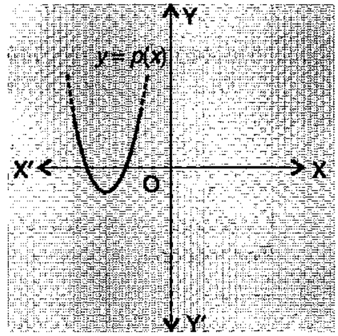
(*i*)



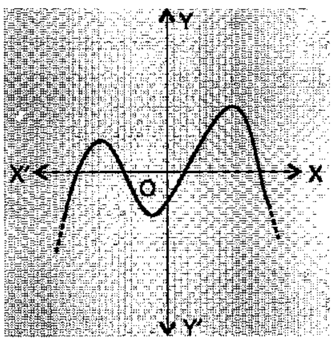
(*ii*)



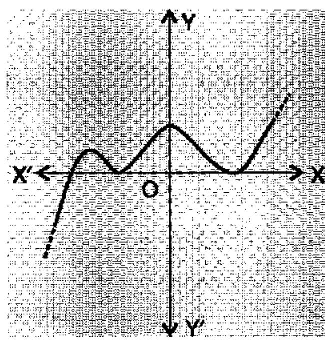
(*iii*)



(*iv*)



(*v*)



(*vi*)

Sol. (*i*) As the graph of polynomial does not meet x-axis, so the polynomial has no zeroes.

(*ii*) As the graph of polynomial cuts (meets) x-axis only once, so the polynomial has exactly one zero.

(*iii*) As the graph of polynomial cuts (meets) x-axis thrice, so the polynomial has three zeroes.

(*iv*) As the graph of polynomial cuts (meets) x-axis twice, so the polynomial has exactly two zeroes.

(*v*) As the graph of polynomial cuts (meets) x-axis four times, so the polynomial has four zeroes.

(*vi*) As the graph of polynomial cuts (meets) x-axis three times, so the polynomial has three zeroes.

Exercise 2.2 *(Page – 33)*

1. *Find the zeroes of the following quadratic polynomials and verify the relationship between the zeroes and the coefficients.*

(i) $x^2 - 2x - 8$ (ii) $4s^2 - 4s + 1$ (iii) $6x^2 - 3 - 7x$

(iv) $4u^2 + 8u$ (v) $t^2 - 15$ (vi) $3x^2 - x - 4$

Sol. (i) Consider polynomial $x^2 - 2x - 8 = (x - 4)(x + 2)$

For zeroes, $x - 4 = 0, x + 2 = 0$

$\Rightarrow x = 4, -2$

Zeroes of the polynomial are 4 and – 2.

$$\text{Sum of zeroes} = 4 + (-2) = 2$$

$$= \frac{-(-2)}{1} = \frac{-\text{Coefficient of } x}{\text{Coefficient of } x^2}$$

$$\text{Product of zeroes} = 4 \times (-2) = -8$$

$$= \frac{-8}{1} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$$

Hence verified.

(ii) Consider polynomial $4s^2 - 4s + 1 = (2s - 1)^2$

For zeroes, $4s^2 - 4s + 1 = 0$

$\therefore \quad (2s - 1)^2 = 0$

$\Rightarrow \quad 2s - 1 = 0 \Rightarrow s = \frac{1}{2}.$

$\therefore$ Polynomial has equal zeroes, *i.e.*, $\frac{1}{2}$ and $\frac{1}{2}$.

$$\text{Sum of zeroes} = \frac{1}{2} + \frac{1}{2} = 1 = \frac{4}{4}$$

$$= -\frac{(-4)}{4} = -\frac{\text{Coefficient of } s}{\text{Coefficient of } s^2}$$

$$\text{Product of zeroes} = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} = \frac{\text{Constant term}}{\text{Coefficient of } s^2}.$$

Hence verified.

(iii) Consider polynomial $6x^2 - 3 - 7x = 6x^2 - 7x - 3$

$= 6x^2 - 9x + 2x - 3 = 3x(2x - 3) + 1(2x - 3)$

$= (2x - 3)(3x + 1)$

For zeroes, $2x - 3 = 0$, $3x + 1 = 0$.

$\Rightarrow \quad x = \frac{3}{2}, -\frac{1}{3}$

$\Rightarrow$ Zeroes of polynomial are $\frac{3}{2}$ and $-\frac{1}{3}$.

Sum of zeroes $= \frac{3}{2} - \frac{1}{3} = \frac{7}{6} = \frac{-(-7)}{6}$

$= -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

Product of zeroes $= \frac{3}{2} \times \frac{(-1)}{3}$

$= \frac{-1}{2} = \frac{-3}{6} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$.

Hence verified.

(*iv*) Consider polynomial $4u^2 + 8u = 4u(u + 2)$.

For zeroes, $\quad 4u(u + 2) = 0$

$\Rightarrow \quad u = 0$ or $u + 2 = 0$

$\therefore$ Zeroes of the polynomial are 0 and -2.

Sum of zeroes $= 0 + (-2) = -2 = \frac{-8}{4}$

$= \frac{-\text{Coefficient of } u}{\text{Coefficient of } u^2}$

Product of zeroes $= 0 \times (-2) = 0 = \frac{0}{4}$

$= \frac{\text{Constant term}}{\text{Coefficient of } u^2}$

Hence verified.

(*v*) Consider polynomial $\quad t^2 - 15 = (t - \sqrt{15})(t + \sqrt{15})$

For zeroes, $(t - \sqrt{15})(t + \sqrt{15}) = 0$

$\Rightarrow \quad t - \sqrt{15} = 0, t + \sqrt{15} = 0$

$\Rightarrow \quad t = \sqrt{15}, \quad t = -\sqrt{15}$

$\therefore$ Zeroes of the polynomial are $\sqrt{15}$ and $-\sqrt{15}$.

Sum of zeroes $= \sqrt{15} + (-\sqrt{15}) = 0$

$$= -\frac{0}{1} = -\frac{\text{Coefficient of } t}{\text{Coefficient of } t^2}$$

Product of zeroes $= (\sqrt{15})(-\sqrt{15}) = -15 = \frac{-15}{1}$

$$= \frac{\text{Constant term}}{\text{Coefficient of } t^2}$$ **Hence verified.**

(*vi*) Consider polynomial $3x^2 - x - 4 = 3x^2 - 4x + 3x - 4$

$= x(3x - 4) + 1(3x - 4) = (x + 1)(3x - 4)$

For zeroes, $(x + 1)(3x - 4) = 0$

$\Rightarrow$ $x + 1 = 0,\ 3x - 4 = 0$

$\Rightarrow$ $x = -1, \frac{4}{3}$

$\therefore$ Zeroes of the polynomial are -1 and $\frac{4}{3}$.

Sum of zeroes $= -1 + \frac{4}{3} = \frac{1}{3} = -\frac{(-1)}{3}$

$$= -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}.$$

Product of zeroes $= (-1)\left(\frac{4}{3}\right) = \frac{-4}{3}$

$$= \frac{\text{Constant term}}{\text{Coefficient of } x^2}.$$

Hence verified.

2. *Find a quadratic polynomial each with the given numbers as the sum and product of its zeroes respectively.*

(*i*) $\frac{1}{4}, -1$ (*ii*) $\sqrt{2}, \frac{1}{3}$ (*iii*) $0, \sqrt{5}$

(*iv*) $1, 1$ (*v*) $-\frac{1}{4}, \frac{1}{4}$ (*vi*) $4, 1$.

Sol. (*i*) Let polynomial be $f(x) = ax^2 + bx + c$ ...(*i*)

Sum of zeroes $= \frac{1}{4} = -\frac{(-1)}{4} = -\frac{b}{a}$ ...(ii)

Product of zeroes $= -1 = \frac{-4}{4} = \frac{c}{a}$ ...(iii)

From equations (ii) and (iii), we get

$$a = 4,\ b = -1,\ c = -4.$$

Substituting these values in equation (i), we get

Polynomial $f(x) = 4x^2 - x - 4$.

We can have infinite such polynomials as $f(x) = k(4x^2 - x - 4)$, k is a real number.

(ii) Let polynomial be $f(x) = ax^2 + bx + c$ ...(i)

Sum of zeroes $= \sqrt{2} = -\frac{(-3\sqrt{2})}{3}$

$= -\frac{b}{a}$ ...(ii)

Product of zeroes $= \frac{1}{3} = \frac{c}{a}$ ...(iii)

From equations (ii) and (iii), we get

$$a = 3,\ b = -3\sqrt{2},\ c = 1.$$

Substituting these values in equation (i), we get

Polynomial $f(x) = 3x^2 - 3\sqrt{2}\,x + 1$.

or $f(x) = k(3x^2 - 3\sqrt{2}\,x + 1)$, k is a real number.

(iii) Let polynomial be $f(x) = ax^2 + bx + c$ ...(i)

Sum of zeroes $= 0 = -\frac{(-0)}{1} = -\frac{b}{a}$ ...(ii)

Product of zeroes $= \sqrt{5} = \frac{\sqrt{5}}{1} = \frac{c}{a}$ ...(iii)

From equations (ii) and (iii), we get

$$a = 1,\ b = 0,\ c = \sqrt{5}$$

Substituting these values in equation (i), we get

Polynomial $f(x) = x^2 + \sqrt{5}$.

or $f(x) = k(x^2 + \sqrt{5})$, k is a real number.

(*iv*) Let polynomial be $f(x) = ax^2 + bx + c$ ...(*i*)

Sum of zeroes $= 1 = \frac{1}{1} = -\frac{(-1)}{1}$

$= -\frac{b}{a}$ ...(*ii*)

Product of zeroes $= 1 = \frac{1}{1} = \frac{c}{a}$ ...(*iii*)

From equations (*ii*) and (*iii*), we get

$a = 1, b = -1, c = 1.$

Substituting these values in equation (*i*), we get

Polynomial $f(x) = x^2 - x + 1.$

or $f(x) = k(x^2 - x + 1)$, k is a real number.

(*v*) Let polynomial be $f(x) = ax^2 + bx + c$ ...(*i*)

Sum of zeroes $= -\frac{1}{4} = -\frac{1}{4} = -\frac{b}{a}$ ...(*ii*)

Product of zeroes $= \frac{1}{4} = \frac{c}{a}$ ...(*iii*)

From equations (*ii*) and (*iii*), we get

$a = 4, b = 1, c = 1$

Substituting these values in equation (*i*), we get

Polynomial $f(x) = 4x^2 + x + 1.$

or $f(x) = k(4x^2 + x + 1)$, k is a real number.

(*vi*) Let polynomial be $f(x) = ax^2 + bx + c$ ...(*i*)

Sum of zeroes $= 4 = -\frac{(-4)}{1} = -\frac{b}{a}$ ...(*ii*)

Product of zeroes $= 1 = \frac{1}{1} = \frac{c}{a}$ ...(*iii*)

From equations (*ii*) and (*iii*), we get

$a = 1, b = -4, c = 1$

Substituting these values in equation (*i*), we get

Polynomial $f(x) = x^2 - 4x + 1.$

or $f(x) = k(x^2 - 4x + 1)$, k is a real number.

Exercise 2.3 *(Page – 36)*

1. *Divide the polynomial p(x) by the polynomial g(x) and find the quotient and remainder in each of the following:*

(*i*) $p(x) = x^3 - 3x^2 + 5x - 3$, $g(x) = x^2 - 2$

(*ii*) $p(x) = x^4 - 3x^2 + 4x + 5$, $g(x) = x^2 + 1 - x$

(*iii*) $p(x) = x^4 - 5x + 6$, $g(x) = 2 - x^2$.

Sol. (*i*) $p(x) = x^3 - 3x^2 + 5x - 3$ and $g(x) = x^2 - 2$

$$\begin{array}{r|l} & x - 3 \\ \hline x^2 - 2 & x^3 - 3x^2 + 5x - 3 \\ & x^3 \quad\quad - 2x \\ & - \quad\quad\quad + \\ \hline & -3x^2 + 7x - 3 \\ & -3x^2 \quad\quad + 6 \\ & + \quad\quad\quad - \\ \hline & \quad\quad 7x - 9 \end{array}$$

First term of quotient is $\frac{x^3}{x^2} = x$

Second term of quotient is $\frac{-3x^2}{x^2} = -3$

We have quotient $q(x) = x - 3$ and remainder $r(x) = 7x - 9$.

(*ii*) $p(x) = x^4 - 3x^2 + 4x + 5$, $g(x) = x^2 + 1 - x$.

$$\begin{array}{r|l} & x^2 + x - 3 \\ \hline x^2 - x + 1 & x^4 - 3x^2 + 4x + 5 \\ & x^4 + x^2 - x^3 \\ & - \quad - \quad + \\ \hline & x^3 - 4x^2 + 4x + 5 \\ & x^3 - x^2 + x \\ & - \quad + \quad - \\ \hline & -3x^2 + 3x + 5 \\ & -3x^2 + 3x - 3 \\ & + \quad - \quad + \\ \hline & \quad\quad\quad 8 \end{array}$$

First term of quotient $= \frac{x^4}{x^2} = x^2$

Second term of quotient $= \frac{x^3}{x^2} = x$

Third term of quotient $= \frac{-3x^2}{x^2} = -3$

$\therefore$ Quotient $q(x) = x^2 + x - 3$; remainder $r(x) = 8$.

(*iii*) $p(x) = x^4 - 5x + 6$, $g(x) = 2 - x^2$

$$
\begin{array}{r|l}
 & -x^2 - 2 \\ \hline
-x^2 + 2 & x^4 - 5x + 6 \\
 & x^4 \quad\quad - 2x^2 \\
 & - \quad\quad + \\ \hline
 & 2x^2 - 5x + 6 \\
 & 2x^2 \quad\quad - 4 \\
 & - \quad\quad + \\ \hline
 & -5x + 10
\end{array}
$$

First term of quotient $= \frac{x^4}{-x^2} = -x^2$

Second term of quotient $= \frac{2x^2}{-x^2} = -2$

$\therefore$ Quotient $q(x) = -x^2 - 2$,

remainder $r(x) = -5x + 10$.

2. *Check whether the first polynomial is a factor of the second polynomial by dividing the second polynomial by the first polynomial:*

(*i*) $t^2 - 3,\ 2t^4 + 3t^3 - 2t^2 - 9t - 12$

(*ii*) $x^2 + 3x + 1,\ 3x^4 + 5x^3 - 7x^2 + 2x + 2$

(*iii*) $x^3 - 3x + 1,\ x^5 - 4x^3 + x^2 + 3x + 1.$

Sol. (*i*) Let $p(t) = 2t^4 + 3t^3 - 2t^2 - 9t - 12$, $g(t) = t^2 - 3$

Let us divide $p(t)$ by $g(t)$

$$
\begin{array}{r|l}
 & 2t^2 + 3t + 4 \\ \hline
t^2 - 3 & 2t^4 + 3t^3 - 2t^2 - 9t - 12 \\
 & 2t^4 \quad\quad - 6t^2 \\
 & - \quad\quad + \\ \hline
 & 3t^3 + 4t^2 - 9t - 12 \\
 & 3t^3 \quad\quad - 9t \\
 & - \quad\quad + \\ \hline
 & 4t^2 - 12 \\
 & 4t^2 - 12 \\
 & - \quad + \\ \hline
 & 0
\end{array}
$$

First term of quotient $= \frac{2t^4}{t^2} = 2t^2$

Second term of quotient $= \frac{3t^3}{t^2} = 3t$

Third term of quotient $= \frac{4t^2}{t^2} = 4$

Here, quotient $q(t) = 2t^2 + 3t + 4$, remainder $r(t) = 0$.
As remainder is 0. Hence, $t^2 - 3$ is a factor of the polynomial

$$2t^4 + 3t^3 - 2t^2 - 9t - 12.$$

(*ii*) Let us divide second polynomial

$p(x) = 3x^4 + 5x^3 - 7x^2 + 2x + 2$ by $q(x) = x^2 + 3x + 1$.

$$
\begin{array}{r|l}
 & 3x^2 - 4x + 2 \\
\hline
x^2 + 3x + 1 & 3x^4 + 5x^3 - 7x^2 + 2x + 2 \\
 & 3x^4 + 9x^3 + 3x^2 \\
 & - \quad - \quad - \\
\hline
 & -4x^3 - 10x^2 + 2x + 2 \\
 & -4x^3 - 12x^2 - 4x \\
 & + \quad + \quad + \\
\hline
 & 2x^2 + 6x + 2 \\
 & 2x^2 + 6x + 2 \\
 & - \quad - \quad - \\
\hline
 & 0
\end{array}
$$

First term of quotient $= \frac{3x^4}{x^2} = 3x^2$

Second term of quotient $= \frac{-4x^3}{x^2} = -4x$

Third term of quotient $= \frac{2x^2}{x^2} = 2$

Here, quotient $q(x) = 3x^2 - 4x + 2$,

remainder $r(x) = 0$

We have $3x^4 + 5x^3 - 7x^2 + 2x + 2$

$= (x^2 + 3x + 1)(3x^2 - 4x + 2) + 0$

As remainder is zero. Hence, first polynomial is a factor of the second polynomial.

(*iii*) Let $p(x) = x^5 - 4x^3 + x^2 + 3x + 1$, $g(x) = x^3 - 3x + 1$

Let us divide $p(x)$ by $q(x)$.

$$
\begin{array}{r|l}
 & x^2 - 1 \\
\hline
x^3 - 3x + 1 & x^5 - 4x^3 + x^2 + 3x + 1 \\
 & x^5 - 3x^3 + x^2 \\
 & - \quad + \quad - \\
\hline
 & -x^3 + 3x + 1 \\
 & -x^3 + 3x - 1 \\
 & + \quad - \quad + \\
\hline
 & 2
\end{array}
$$

First term of quotient $= \frac{x^5}{x^3} = x^2$

Second term of quotient $= \frac{-x^3}{x^3} = -1$

Here, quotient $q(x) = x^2 - 1$, remainder $r(x) = 2$

As remainder is not zero.

Hence, $x^3 - 3x + 1$ is not a factor of $x^5 - 4x^3 + x^2 + 3x + 1$.

3. *Obtain all other zeroes of $3x^4 + 6x^3 - 2x^2 - 10x - 5$, if two of its zeroes are $\sqrt{\frac{5}{3}}$ and $-\sqrt{\frac{5}{3}}$.*

Sol. Two zeroes of polynomial $3x^4 + 6x^3 - 2x^2 - 10x - 5$ are

$$x = \sqrt{\frac{5}{3}} \text{ and } x = -\sqrt{\frac{5}{3}}$$

$\therefore$ $(\sqrt{3}\,x - \sqrt{5})$ and $(\sqrt{3}\,x + \sqrt{5})$ are factors of the polynomial

$3x^4 + 6x^3 - 2x^2 - 10x - 5$

$\Rightarrow$ $(\sqrt{3}\,x - \sqrt{5})(\sqrt{3}\,x + \sqrt{5}) = 3x^2 - 5$ is a factor of the polynomial

$3x^4 + 6x^3 - 2x^2 - 10x - 5$

Let us use division algorithm to find other zeroes.

Dividing $3x^4 + 6x^3 - 2x^2 - 10x - 5$ by $(3x^2 - 5)$

$$
\begin{array}{r|l}
 & x^2 + 2x + 1 \\
\hline
3x^2 - 5 & 3x^4 + 6x^3 - 2x^2 - 10x - 5 \\
 & 3x^4 \qquad\quad - 5x^2 \\
 & - \qquad\qquad + \\
\hline
 & 6x^3 + 3x^2 - 10x - 5 \\
 & 6x^3 \qquad\quad - 10x \\
 & - \qquad\qquad + \\
\hline
 & 3x^2 - 5 \\
 & 3x^2 - 5 \\
 & - \quad + \\
\hline
 & 0
\end{array}
$$

First term of quotient $= \dfrac{3x^4}{3x^2} = x^2$

Second term of quotient $= \dfrac{6x^3}{3x^2} = 2x$

Third term of quotient $= \dfrac{3x^2}{3x^2} = 1$

By division algorithm, we have

$$3x^4 + 6x^3 - 2x^2 - 10x - 5 = (3x^2 - 5)(x^2 + 2x + 1)$$
$$= (3x^2 - 5)(x + 1)^2$$

Other zeroes of the polynomial are $-1, -1$.

[By using $x + 1 = 0$]

Hence, zeroes of the polynomial

$$3x^4 + 6x^3 - 2x^2 - 10x - 5 \text{ are}$$

$$\sqrt{\frac{5}{3}}, -\sqrt{\frac{5}{3}}, -1 \text{ and } -1.$$

4. *On dividing $x^3 - 3x^2 + x + 2$ by a polynomial g(x), the quotient and remainder were $x - 2$ and $-2x + 4$, respectively. Find g(x).*

Sol. We have $p(x) = x^3 - 3x^2 + x + 2$, $g(x)$,

$$q(x) = x - 2 \text{ and } r(x) = -2x + 4.$$

Using division algorithm, we have

$$p(x) = g(x) \times q(x) + r(x)$$

$$\Rightarrow \quad x^3 - 3x^2 + x + 2 = g(x) \times (x - 2) + (-2x + 4)$$

$$\Rightarrow \quad x^3 - 3x^2 + x + 2 + 2x - 4 = g(x) \times (x - 2)$$

$$\Rightarrow \quad g(x) \times (x - 2) = x^3 - 3x^2 + 3x - 2$$

$$g(x) = \frac{x^3 - 3x^2 + 3x - 2}{x - 2}$$

$$\begin{array}{r|l} & x^2 - x + 1 \\ \hline x - 2 & x^3 - 3x^2 + 3x - 2 \\ & x^3 - 2x^2 \\ & - \quad + \\ \hline & -x^2 + 3x - 2 \\ & -x^2 + 2x \\ & + \quad - \\ \hline & x - 2 \\ & x - 2 \\ & - \quad + \\ \hline & 0 \end{array}$$

First term of quotient

$$= \frac{x^3}{x} = x^2$$

Second term of quotient

$$= \frac{-x^2}{x} = -x$$

Third term of quotient

$$= \frac{x}{x} = 1$$

Hence, $g(x) = x^2 - x + 1$.

5. *Give examples of polynomials p(x), g(x), q(x) and r(x), which satisfy the division algorithm and*

(*i*) *deg p(x) = deg q(x)* (*ii*) *deg q(x) = deg r(x)*

(*iii*) *deg r(x) = 0*

Sol. (*i*) Let $p(x) = 3x^2 + 6x - 11$ and $g(x) = 3$

Then $q(x) = x^2 + 2x - 3$, $r(x) = -2$

Here, $\deg p(x) = \deg q(x)$

(*ii*) Let $p(x) = x^3 + 6x^2 + 5x$ and $g(x) = x^2 + 2$

Then $q(x) = x + 6$, $r(x) = -x - 12$

Here, $\deg q(x) = \deg r(x)$.

(*iii*) Let $p(x) = 3x^3 + 5x^2 - 6x + 7$ and $g(x) = x - 1$

Then $q(x) = 3x^2 + 8x + 2$, $r(x) = 9$

Here, $\deg r(x) = 0$

Note: Each of (*i*), (*ii*) and (*iii*) has several examples.

Exercise 2.4 *(Page – 36-37)*

1. *Verify that the numbers given alongside of the cubic polynomials below are their zeroes. Also verify the relationship between the zeroes and the coefficients in each case:*

(*i*) $2x^3 + x^2 - 5x + 2;\ \frac{1}{2}, 1, -2$

(*ii*) $x^3 - 4x^2 + 5x - 2;\ 2, 1, 1$

Sol. (*i*) Let $p(x) = 2x^3 + x^2 - 5x + 2$

If $\frac{1}{2}$, 1, – 2 are zeroes of $p(x)$, then

$p\left(\frac{1}{2}\right) = 0$, $p(1) = 0$ and $p(-2) = 0$.

Let us verify.

$$p\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 - 5 \times \frac{1}{2} + 2$$

$$= \frac{2}{8} + \frac{1}{4} - \frac{5}{2} + 2$$

$$= \frac{2+2-20+16}{8} = \frac{0}{8} = 0.$$

$p(1) = 2(1)^3 + (1)^2 - 5(1) + 2 = 2 + 1 - 5 + 2 = 0.$

$p(-2) = 2(-2)^3 + (-2)^2 - 5 \times (-2) + 2$

$= -16 + 4 + 10 + 2 = 0$

Hence, we can say $\alpha = \frac{1}{2}$, $\beta = 1$, $\gamma = -2$ are zeroes of $p(x)$.

Relationship:

$$\alpha + \beta + \gamma = \frac{1}{2} + 1 - 2 = \frac{1}{2} - 1 = -\frac{1}{2}$$

$$= \frac{-\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{1}{2} \times 1 + 1 \times (-2) + (-2) \times \frac{1}{2}$$

$$= \frac{1}{2} + (-2) - 1 = -\frac{5}{2} = \frac{-5}{2}$$

$$= \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$$

and $\alpha\beta\gamma = \frac{1}{2} \times 1 \times (-2) = -1 = -\frac{(2)}{2}$

$$= -\frac{\text{Constant term}}{\text{Coefficient of } x^3}.$$

Hence, relationship is verified.

(*ii*) Let $q(x) = x^3 - 4x^2 + 5x - 2$

If 2, 1 and 1 are zeroes of $q(x)$, then $q(2) = 0$ and $q(1) = 0$.

Let us verify.

Now $q(2) = (2)^3 - 4(2)^2 + 5(2) - 2 = 0 = 8 - 16 + 10 - 2 = 0$

$q(1) = (1)^3 - 4(1)^2 + 5(1) - 2 = 1 - 4 + 5 - 2 = 0$

Hence, verified.

Let $\alpha = 2$, $\beta = 1$, $\gamma = 1$.

Relationship:

Sum of zeroes $= \alpha + \beta + \gamma = 2 + 1 + 1 = 4$

$$= -\frac{(-4)}{1} = -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

Sum of product of zeroes taken in pair

$$= \alpha\beta + \beta\gamma + \gamma\alpha = 2 + 1 + 2 = 5$$

$$= \frac{5}{1} = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$$

$$\text{Product of zeroes} = \alpha\beta\gamma = 2 = \frac{2}{1} = -\frac{(-2)}{1}$$

$$= -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

Hence, relationship is verified.

2. *Find a cubic polynomial with the sum, sum of the product of its zeroes taken two at a time, and the product of its zeroes as 2, – 7, – 14 respectively.*

Sol. Let polynomial be $f(x) = ax^3 + bx^2 + cx + d$ ...(*i*)

Let α, β and γ be the zeroes of the polynomial

Given, $\alpha + \beta + \gamma = 2 = \frac{-(-2)}{1} = -\frac{b}{a}$ ...(*ii*)

$$\alpha\beta + \beta\gamma + \gamma\alpha = -7 = \frac{-7}{1} = \frac{c}{a} \quad ...(iii)$$

$$\alpha\beta\gamma = -14 = -\frac{14}{1} = -\frac{d}{a} \quad ...(iv)$$

From (*ii*), (*iii*) and (*iv*), we have

$$a = 1, b = -2, c = -7, d = 14.$$

Substituting these values in (*i*), we get

$$f(x) = x^3 - 2x^2 - 7x + 14$$

or $f(x) = k(x^3 - 2x^2 - 7x + 14)$,

where k is a real number.

3. *If the zeroes of the polynomial $x^3 - 3x^2 + x + 1$ are $a - b$, a, $a + b$, find a and b.*

Sol. Let the given polynomial be $Ax^3 + Bx^2 + Cx + D$

Here, A = 1, B = – 3, C = 1, D = 1

Zeroes are $a - b$, a and $a + b$.

$$\text{Sum of zeroes} = -\frac{B}{A}$$

$\Rightarrow \quad a - b + a + a + b = 3$

$\Rightarrow \quad 3a = 3 \quad \Rightarrow \quad a = 1.$

$$\text{Product of zeroes} = -\frac{D}{A}$$

$\Rightarrow \quad (a - b)\, a\, (a + b) = -\frac{1}{1}$

$\Rightarrow \quad (1 - b) . 1 . (1 + b) = -1$

$\Rightarrow \quad 1 - b^2 = -1$

$\Rightarrow \quad b^2 = 2 \quad \Rightarrow \quad b = \pm\sqrt{2}$

Hence, $a = 1$, $b = \pm\sqrt{2}$.

4. *If two zeroes of the polynomial $x^4 - 6x^3 - 26x^2 + 138x - 35$ are $2 \pm \sqrt{3}$, find other zeroes.*

Sol. Given polynomial $p(x) = x^4 - 6x^3 - 26x^2 + 138x - 35$

As two zeroes are $x = 2 \pm \sqrt{3}$.

So, $\{x - (2 + \sqrt{3})\}\{x - (2 - \sqrt{3})\}$ is a factor of $p(x)$.

i.e., $(x^2 - 4x + 1)$ is a factor of $p(x)$.

$$
\begin{array}{r|l}
 & x^2 - 2x - 35 \\
\hline
x^2 - 4x + 1 & x^4 - 6x^3 - 26x^2 + 138x - 35 \\
 & x^4 - 4x^3 + x^2 \\
 & - \quad + \quad - \\
\hline
 & -2x^3 - 27x^2 + 138x - 35 \\
 & -2x^3 + 8x^2 - 2x \\
 & + \quad - \quad + \\
\hline
 & -35x^2 + 140x - 35 \\
 & -35x^2 + 140x - 35 \\
 & + \quad - \quad + \\
\hline
 & 0
\end{array}
$$

First term of quotient $= \dfrac{x^4}{x^2} = x^2$

Second term of quotient $= \dfrac{-2x^3}{x^2} = -2x$

Third term of quotient $= \dfrac{-35x^2}{x^2} = -35$

$\therefore \quad p(x) = (x^2 - 4x + 1)(x^2 - 2x - 35)$

For other zeroes, $x^2 - 2x - 35 = 0$.

$\Rightarrow \quad x^2 - 7x + 5x - 35 = 0$

$\Rightarrow \quad x(x - 7) + 5(x - 7) = 0$

$\Rightarrow \quad (x + 5)(x - 7) = 0$

$\Rightarrow \quad x + 5 = 0, x - 7 = 0$

$\Rightarrow \quad x = -5, 7$

Hence, other zeroes are -5 and 7.

5. *If the polynomial $x^4 - 6x^3 + 16x^2 - 25x + 10$ is divided by another polynomial $x^2 - 2x + k$, the remainder comes out to be $x + a$, find k and a.*

Sol. When $x^4 - 6x^3 + 16x^2 - 25x + 10$ is divided by $x^2 - 2x + k$, remainder is $x + a$.

Using division algorithm,

$x^4 - 6x^3 + 16x^2 - 25x + 10 = (x^2 - 2x + k)\,q(x) + (x + a)$,

where $q(x)$ is quotient.

$\Rightarrow \quad x^4 - 6x^3 + 16x^2 - 26x + (10 - a) = (x^2 - 2x + k)\,q(x)$

$\Rightarrow x^2 - 2x + k$ is a factor of $x^4 - 6x^3 + 16x^2 - 26x + (10 - a)$.

$$
\begin{array}{r|l}
 & x^2 - 4x + (8 - k) \\ \hline
x^2 - 2x + k & x^4 - 6x^3 + 16x^2 - 26x + (10 - a) \\
 & x^4 - 2x^3 + kx^2 \\
 & -\quad +\quad - \\ \hline
 & -4x^3 + (16 - k)x^2 - 26x + (10 - a) \\
 & -4x^3 + 8x^2 - 4kx \\
 & +\quad -\quad + \\ \hline
 & (8 - k)x^2 - (26 - 4k)x + (10 - a) \\
 & (8 - k)x^2 - (16 - 2k)x + (8 - k)k \\
 & -\quad +\quad - \\ \hline
 & (-10 + 2k)x + (k^2 - 8k + 10 - a)
\end{array}
$$

First term of quotient $= \dfrac{x^4}{x^2} = x^2$

Second term of quotient $= \dfrac{-4x^3}{x^2} = -4x$

Third term of quotient $= \dfrac{(8 - k)x^2}{x^2} = (8 - k)$

As remainder is zero,

$\therefore \quad (-10 + 2k)x + (k^2 - 8k + 10 - a) = 0$

$\therefore \quad -10 + 2k = 0$ and $k^2 - 8k + 10 - a = 0$

$\Rightarrow \quad k = 5$ and $25 - 40 + 10 - a = 0$

$\Rightarrow \quad k = 5$ and $a = -5$.